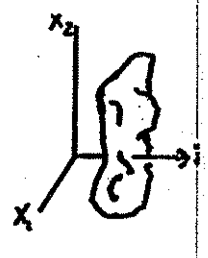
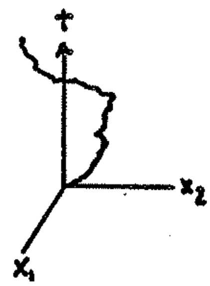
T. Halpin-Healy, Troisième Cycle de la Physique en Suisse Romande, 5 November 1992.

IMPURITY-STRICKEN MAGNETS,



DIRECTED POLYMERS,



EDEN CLUSTERS,

AND ALL THAT.

TIM HALPIN-HEALY

"DIVERSE MANIFOLDS IN RANDOM MEDIA"
PRL 62, 442 (1989)
63, 917 (1989)
PRA (1990)
42, 711

thanks: M. KARDAR (MIT)
Y.-C ZHANG (ROMA)
* T. NATTERMANN (JULICH)
M. FISHER, D. THIRUMALAI (UMD)
J. KRUG (MUNICH)

next time: emphasis on DPRM

* DEPINNING BY QUENCHED RANDOMNESS
* GROUND-STATE INSTABILITIES
* ULTRAMETRIC STRUCTURE OF RANDOM ENERGY LANDSCAPE
* FINITE-TEMPERATURE PHASE TRANSITION
* UNIVERSAL AMPLITUDES
* PROBABILITY DISTRIBUTIONS
* RSB?
* other KPZ-related matters...

RF ISING MODEL

PRB 28, 2588 (1983)

Surface tension, roughening, and lower critical dimension in the random-field Ising model

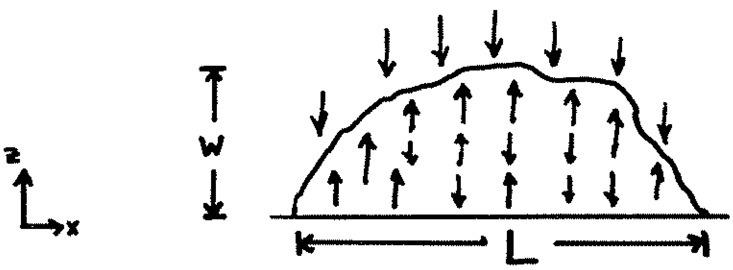
G. Grinstein
IBM Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598

Shang-keng Ma
Department of Physics and Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92093

(Received 3 May 1983)

A continuum interface model is constructed to study the low-temperature properties of domain walls in the random-field Ising model (RFIM). The width of the domain wall and its surface tension are computed by three methods: Simple energy accounting, dimensional arguments, and approximate renormalization-group calculations. All methods yield a surface tension which is positive at sufficiently low temperature for small random fields, h, provided that the dimensionality $d>2$. The lower critical dimension of the RFIM is thus argued to be 2. While effects due to discreteness of a lattice are argued to alter some of the continuum results quantitatively, they do not change these central conclusions.

IMRY-MA DOMAIN WALL ARGUMENT:



ELASTIC COST: $\int d^{d-1}x \, (\nabla z)^2$

RF GAIN

$$L^{d-3} W^2 \sim \left(W L^{d-1}\right)^{\frac{1}{2}} \Rightarrow W \sim L^{\frac{5-d}{3}} \qquad \frac{\epsilon}{3} = \zeta_{RF}$$

"IMPURITY ROUGHENING"

compare to,

$$L^{d-3} W^2 \sim kT \Rightarrow W \sim L^{\frac{3-d}{2} = \zeta_{TH}}$$

"THERMAL ROUGHENING"

typically,

IMPURITY ROUGHENING >> THERMAL ROUGHENING $\longrightarrow$ ZERO TEMPERATURE FIXED POINT

SIMULATION CONFIRMS:

Numerical Evidence for $d_c = 2$ in the Random-Field Ising Model

J. F. Fernandez,[a] G. Grinstein, Y. Imry,[b] and S. Kirkpatrick
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598

(Received 20 May 1983)

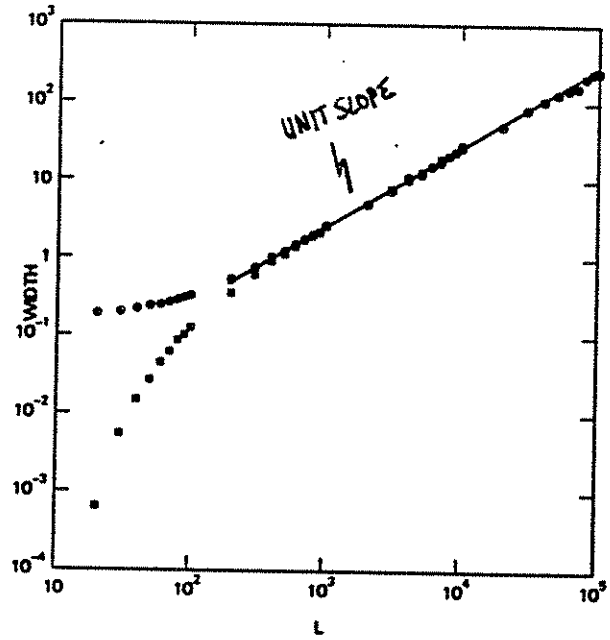
PRL 51, 203 (1983)

A solid-on-solid interface representation of the random-field Ising model is studied numerically in two dimensions. The interface width varies linearly with sample size, in agreement with simple energy-accounting arguments and recent theories which predict that two is the lower critical dimension of the random-field Ising model.

IN $d=2$, IMRY-MA $\Rightarrow$ $\zeta_{RF} = 1$

SO THAT

$$W \sim L$$



SUBSEQUENTLY,

J.Z. IMBRIE (1984) ⟵ RIGOROUS PROOF OF PHASE TRANSITION IN $d=3$

Y.-C. ZHANG (1986) — $\zeta_{RF}(d=2) = 1$, ANALYTIC WORK ON BURGERS' EQUATION

③

RB ISING MODEL (d=2)

Pinning and Roughening of Domain Walls in Ising Systems Due to Random Impurities

PRL 54, 2708 (1985)

David A. Huse and Christopher L. Henley
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
(Received 12 April 1985)

Randomly placed impurities that alter the local exchange couplings, but do not generate random fields or destroy the long-range order, roughen domain walls in Ising systems for dimensionality $\frac{5}{3} < d < 5$. They also pin (localize) the walls in energetically favorable positions. This drastically slows down the kinetics of ordering. The pinned domain wall is a new critical phenomenon governed by a zero-temperature fixed point. For $d=2$, the critical exponents for domain-wall pinning energies and roughness as a function of length scale are estimated from numerically generated ground states.

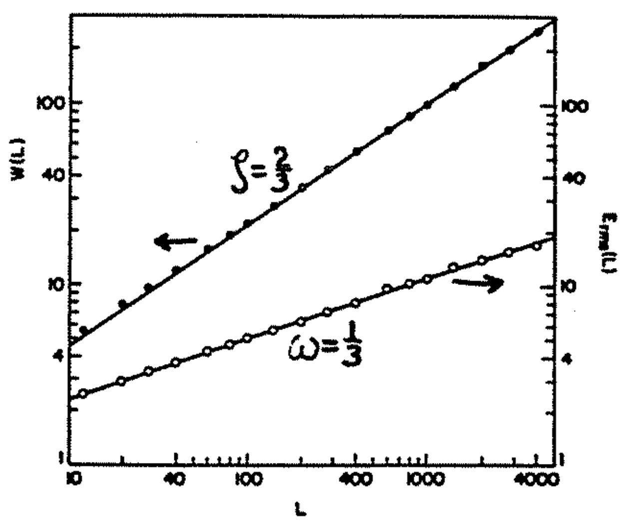


FIG. 1. The root mean square transverse deviation from flatness, $W(L)$, of the ground states of segments of domain wall of length L (upper data), and the energy scale $E_{rms}(L)$ as defined by Eq. (12) (lower data). Fits with straight lines on the log-log plot give exponent estimates $\zeta = 0.66 \pm 0.02$ and $\chi = 0.33 \pm 0.01$, respectively. The straight lines shown here have the slopes given by the conjectured exact exponents $\zeta = \frac{2}{3}$ and $\chi = \frac{1}{3}$. The deviations from a straight line for small L that can be seen in the data for $W(L)$ are presumably due to corrections to scaling. The statistical errors are less than or equal to (for large L) the size of the points on this graph.

HUSE-HENLEY SIMULATIONS

$\Rightarrow \quad \zeta^{RB} = \frac{2}{3}$

ALSO - ENERGY FLUCTUATIONS:

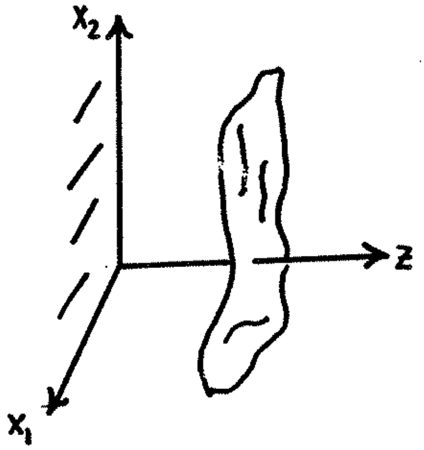
$|\Delta E| \sim L^{\omega}$

But,

$$H = \int dx \left\{ \frac{1}{2}\left(\frac{dz}{dx}\right)^2 + \dots \right\}$$

$$\Rightarrow \quad \underbrace{\omega = 2\zeta - 1}_{\text{SCALING LAW}} = \frac{1}{3}$$

SCALING ARGUMENT (FLORY) $\rightsquigarrow$ DOMAIN WALL ROUGHENING DUE TO QUENCHED CORRELATED DISORDER



$$\mathcal{H} = \int d^{d'=d-1}\vec{x}\,\Big\{ \underbrace{\tfrac{\nu}{2}(\nabla z)^2}_{\text{SURFACE TENSION}} + \sigma \underbrace{\mathcal{V}(z)}_{\substack{\text{RANDOM PINNING POTENTIAL}\\ \int^z dz' h(z') \ \text{RF}}} \Big\}$$

$\langle \mathcal{V} \rangle = 0$

$\langle \mathcal{V}(z,\vec{x})\mathcal{V}(0,0)\rangle = \delta^{d'}(\vec{x})\, z^{-\beta}$

RF : $\beta = -1$ (LINEAR)

RB : $\beta = +1$ (δ-FN)

* ROUGHENING EXPONENT
  $|z| \sim |\vec{x}|^{\zeta(\beta)}$
  AS A FUNCTION OF β.

* SIMPLE RESCALING ARGUMENTS $\Rightarrow$

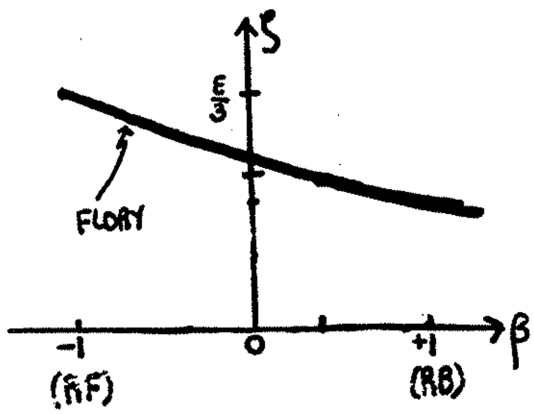
$\vec{x} \to \lambda \vec{x}$ $\qquad y_\nu = 2\zeta + d' - 2$

$z \to \lambda^{\zeta} z$

$\mathcal{V} \to \lambda^{-\frac{1}{2}(d'+\beta\zeta)}\mathcal{V}$ $\qquad y_\sigma = \frac{1}{2}(d' - \beta\zeta)$

* GENERALIZED IMRY-MA $\Rightarrow$

$\zeta_F(\beta) = \dfrac{\varepsilon}{4+\beta}$



NOT QUITE THE WHOLE STORY!

FUNCTIONAL RG (WILSON, FISHER, LIPOWSKY, ...)

starting point

$$\mathcal{Z} = \int_{\substack{\text{ALL}\\\text{CONFIG}}} \mathcal{D}z \; e^{-\int d^{d'}x \left\{ \frac{\gamma}{2}(\nabla z)^2 + V(z) \right\}}$$

$\langle V(z,x) V(z',x') \rangle = \delta^{d'}(x-x') \underbrace{R(z-z')}_{\text{"CORRELATOR"}}$

schematically....

average FREE ENERGY over QUENCHED RANDOMNESS $\Rightarrow$ REPLICA TRICK

$$[f] = [\ln \mathcal{Z}] = \lim_{N\to 0} \left[\frac{e^{N\ln\mathcal{Z}} - 1}{N} \right] = \lim_{N\to 0} \frac{[\mathcal{Z}^N] - 1}{N}$$

consider

$$\mathcal{Z}^N = \int_{\substack{\text{ALL}\\\text{CONFIG}}} \prod_{i=1}^{N} \mathcal{D}z_i \; e^{-\int d^{d'}x \left\{ \frac{\gamma}{2} \sum_{i=1}^{N} (\nabla z_i)^2 + \sum_{i=1}^{N} V(z_i) \right\}}$$

avg $\rightarrow$ $[\bullet] = \int \mathcal{D}V \bullet \, e^{-\frac{1}{2}\int V R^{-1} V}$

INTERACTION AMONGST REPLICAS

$$\Rightarrow \quad [\mathcal{Z}^N] = \int_{\substack{\text{ALL}\\\text{CONFIG}}} \prod_{i=1}^{N} \mathcal{D}z_i \; e^{-\int d^{d'}x \left\{ \frac{\gamma}{2} \sum_{i=1}^{N} (\nabla z_i)^2 - \sum_{i\neq j} R(z_i - z_j) \right\}}$$

calculate,

EFFECTIVE ACTION = TREE LEVEL + ONE-LOOP $\quad \mathrm{Tr}\log(q^2 + R'')$

infinitesimal rescaling $(\lambda = 1 + \delta\ell) \Rightarrow$

$$\frac{\partial R}{\partial \ell} = (\varepsilon - 4\zeta)R + z\zeta R' + \tfrac{1}{2}(R'')^2 - R'' R''(0) + \ldots$$

"FUNCTIONAL" FLOW EQUATION

FIXED POINT FUNCTIONS:

"OLD NEWS" $\quad R^*_{LR} \sim z^{-\beta} \Rightarrow \zeta_F = \frac{\varepsilon}{4+\beta}$

"NEW NEWS" $\quad R^*_{SR} \sim z^{-k} e^{-z^2} \Rightarrow \beta_c = 5 - \varepsilon/3$

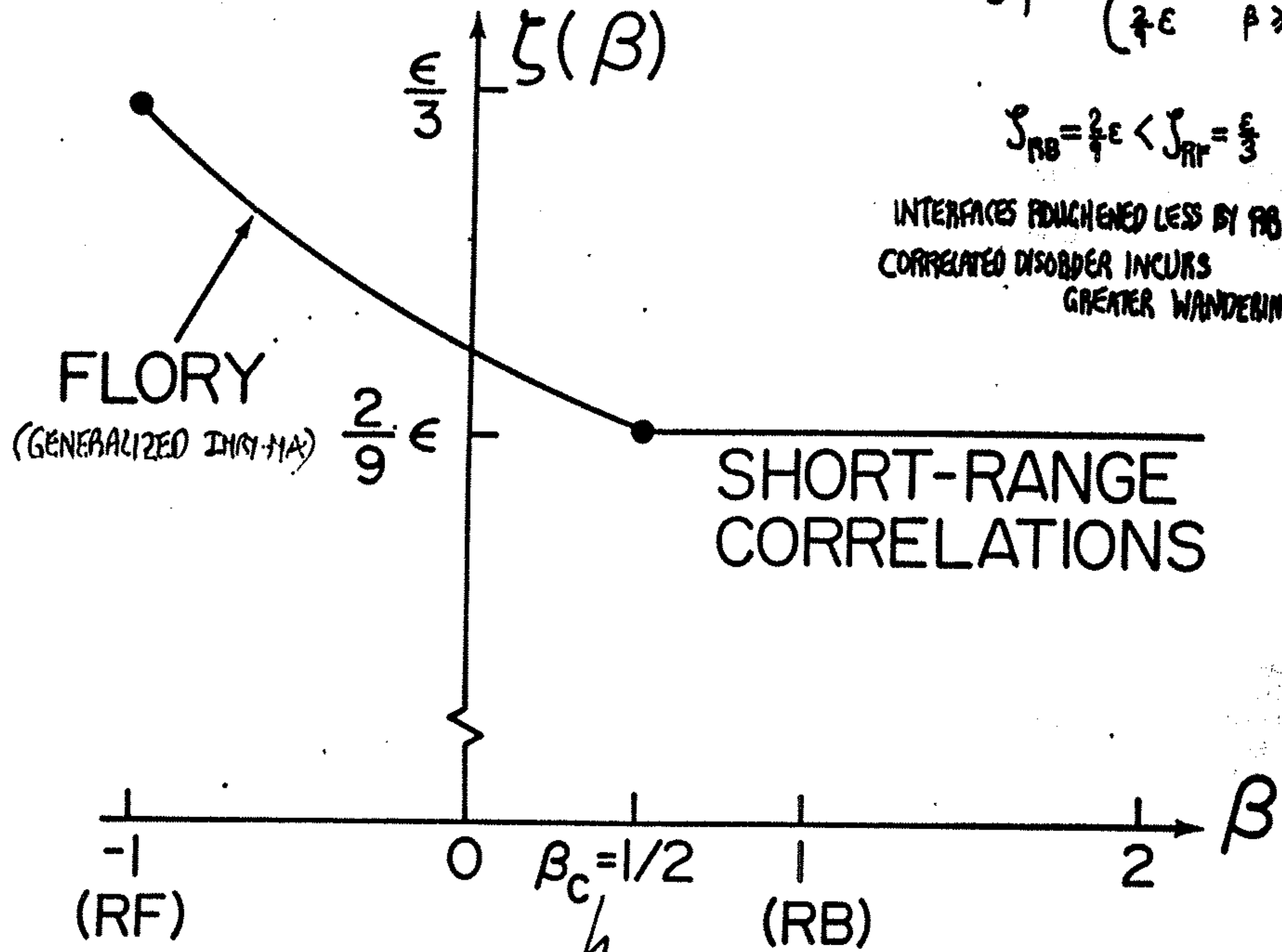
$\zeta_{SR} = \frac{2}{9}\varepsilon \,, \; \beta_c = \frac{1}{2}$ NEW RESULTS

DOMAIN-WALL ROUGHENING EXPONENT:

$$\zeta(\beta) = \begin{cases} \frac{\epsilon}{4+\beta} & \beta \leq \frac{1}{2} \\ \frac{2}{9}\epsilon & \beta \geq \frac{1}{2} \end{cases}$$

$\zeta_{RB} = \frac{2}{9}\epsilon < \zeta_{RF} = \frac{\epsilon}{3}$ — $\zeta_{RB} = \frac{2}{3}$, $\zeta_{RF} = 1$ FOR 1d INTERFACE

INTERFACES ROUGHENED LESS BY RBs
CORRELATED DISORDER INCURS GREATER WANDERING



SR FPF BASIN OF ATTRACTION
NO IMRY-MA FOR RBs
FAILURE OF FLORY MFT

SEE, HOWEVER, ZHANG PRB 42, 4897 (1990)

DIRECTED POLYMER PROBLEM:

IN 1+1, OPTIMAL PATH FLUCTUATIONS SCALE AS,

$|x| \sim t^{2/3}$

$|\Delta E| \sim t^{4/3}$

RB INTERFACE

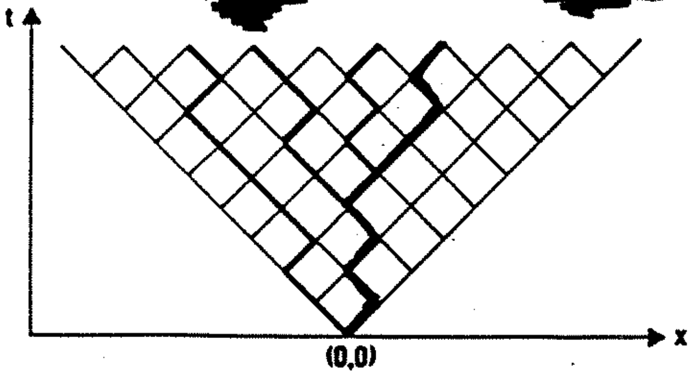
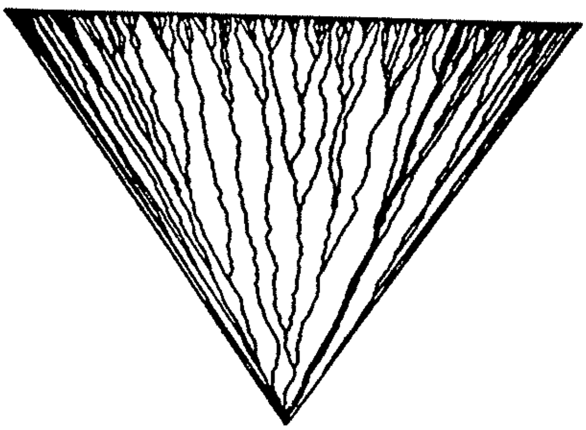


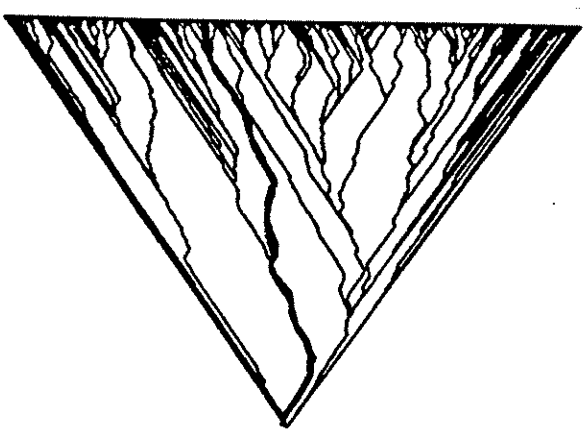
FIG. 1. In this square lattice each bond is given a random value; at the level t there are t points and each of them can be connected to the apex of the triangle in a unique optimal way. A few local optimal paths are shown; one path is the overall best path.

KARDAR + ZHANG
PRL 58, 2087 (1987)

ENSEMBLE OF OPTIMAL ENERGY PATHS:

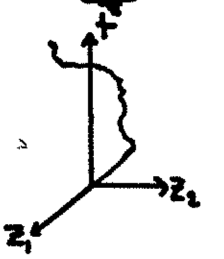


UNCORRELATED RANDOMNESS (RB CASE)



LINEARLY CORRELATED (RF CASE)

MANY-DIMENSIONAL DIRECTED POLYMER



IS $\zeta = \frac{2}{3}$ SUPERUNIVERSAL?

RECALL FOR PURE CASE, $\zeta_0 = \frac{1}{2}$ IS INDEPENDENT OF DIMENSIONALITY

$$\mathcal{H} = \int dt \left\{ \frac{1}{2}\left(\frac{d\vec{z}}{dt}\right)^2 + V(\vec{z}) \right\} \longrightarrow \langle V V \rangle = \delta(t) R(\vec{z})$$

WHERE, $\vec{z}(t) =$ n-COMPONENT VECTOR FIELD

FUNCTIONAL RG $\Rightarrow$

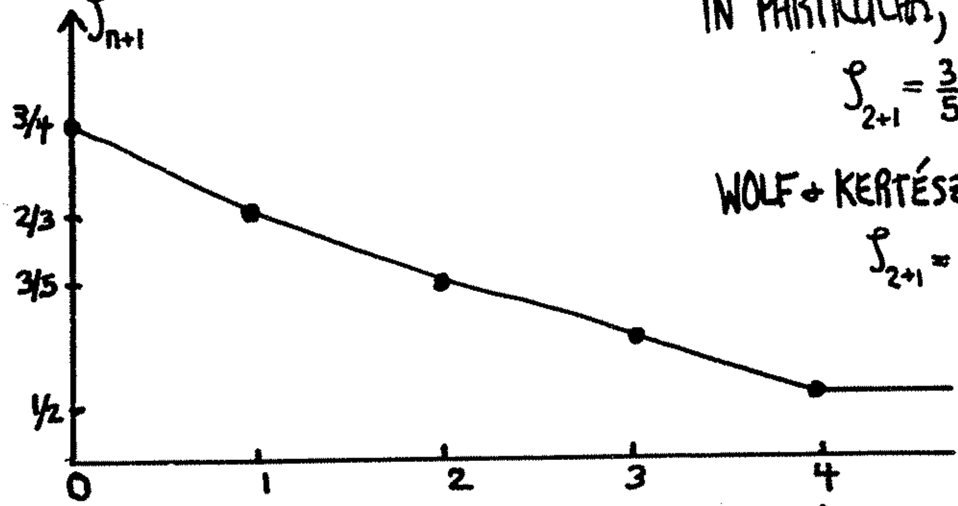
$$\frac{\partial R}{\partial l} = (3-4\zeta)R + z\zeta R' + \frac{1}{2}(R'')^2 - R''R''(0) + \frac{(n-1)}{2}\frac{R'^2}{z} - \frac{(n-1)}{2}\frac{R'}{z}R''(0) + \dots$$

AS BEFORE, SR FPF + FLORY $\Rightarrow$

$$\frac{dT}{dl} = \left(\frac{1}{2} - \zeta_{SR}\right)T$$

$$\boxed{\beta_c = \frac{1}{2}, \quad \zeta_{SR}(n) = \frac{6}{8+n}}$$

NEW RESULT



IN PARTICULAR,

$$\zeta_{2+1} = \frac{3}{5}$$

WOLF + KERTÉSZ (EDEN GROWTH):

$$\zeta_{2+1} = 0.60 \pm 0.02$$

But more recent work of Kim + Kosterlitz gives 0.625 ± 0.002

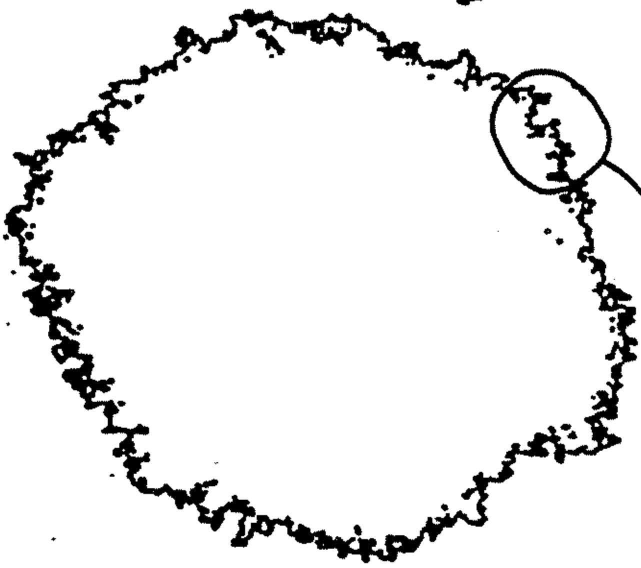
$\frac{5}{8}$?

UPPER CRITICAL DIMENSION? UNLIKELY, SINCE FUNCTIONAL RG IS ONLY APPROXIMATE.

NONEQUILIBRIUM
"KINETIC ROUGHENING"

EDEN CLUSTERS - SIMPLE MODEL OF STOCHASTIC GROWTH

M. EDEN (1961)

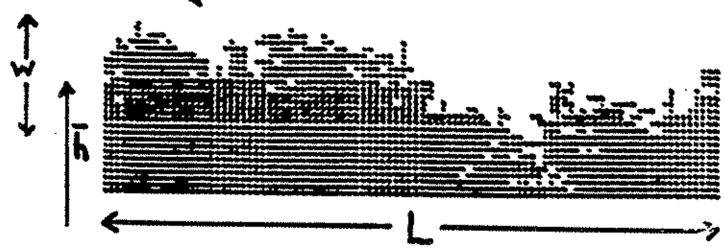


"CIRCULAR GEOMETRY"

* EPIDEMIC PROPAGATION
* SPREADING OF DISEASE

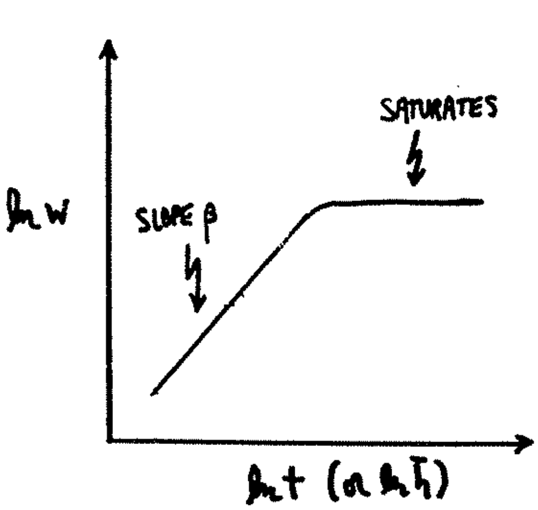
STUDY SCALING PROPERTIES OF THE EDEN SURFACE

ACTIVE ZONE:



"STRIP GEOMETRY"

- SIMULATIONS ⇒ DYNAMIC SCALING BEHAVIOR:



SCALING ANSATZ

STEADY-STATE

DYNAMIC EXPONENT

$$W \simeq L^{\chi}\, \varphi(t/L^{z})$$

WHERE,

$\varphi(x) \to \text{const} \qquad x \to \infty$

$\varphi(x) \to x^{\beta = \frac{\chi}{z}} \qquad x \to 0$

→ ENTER THE REALM OF KPZ, OFFICIALLY:

NONLINEAR LANGEVIN EQUATION

KARDAR, PARISI + ZHANG (1986)
Phys. Rev. Lett 56, 889

$$\frac{\partial h}{\partial t} = \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t)$$

SURFACE TENSION RELAXATION

NONLINEARITY, TIED TO BREAKING OF TIME REVERSAL INVARIANCE

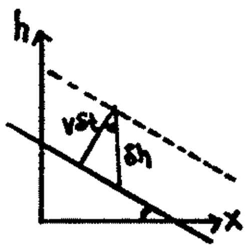
STOCHASTIC NOISE

$\langle \eta \rangle = 0$

$\langle \eta\eta \rangle = \delta(x-x')\,\delta(t-t')$

ORIGIN: COARSE-GRAIN ⇒ GROWTH OCCURS IN DIRECTION LOCALLY ⊥ SURFACE

↑ v



$$\delta h = \sqrt{(v\delta t)^2 + (v\delta t\,\nabla h)^2}$$

$$\Rightarrow$$

$$\dot{h} = v\sqrt{1+(\nabla h)^2} \approx v + \frac{v}{2}(\nabla h)^2$$

* EVAPORATION/DEPOSITION $(v=0)$ - PLISCHKE, RACZ, + LIU (1987) ⟶ TRIVIAL SCALING $(\alpha=0,\ z=2)$

* DIRECTED POLYMER PROBLEM:

$$h = \ln \mathcal{Z} \Rightarrow \partial_t \mathcal{Z} = \left(\nabla^2 + \eta(x,t)\right)\mathcal{Z} \Rightarrow \mathcal{Z}(x,t) = \int \mathcal{D}x\, e^{-\int dt\left\{\frac{1}{2}\dot{x}^2 + \eta\right\}}$$

"SCHRODINGER EQUATION" RANDOM POTENTIAL

PATH INTEGRAL / PARTITION FN

h, f FLUCTUATE SAME WAY ⇒ $\beta = \omega$, ALSO $z = \frac{1}{\zeta}$ ⇒ $z = \frac{3}{2}$, $\chi = \beta z = \frac{1}{2}$

2d EDEN MODEL

* KPZ CRITICAL EXPONENTS: BALLISTIC DEPOSITION

Growth in a Restricted Solid-on-Solid Model

PRL 62, 2289 (1989).

Jin Min Kim and J. M. Kosterlitz

Department of Physics, Brown University, Providence, Rhode Island 02912

(Received 23 January 1989)

Extensive simulations of growth in a stochastic ballistic deposition model on a $(d-1)$-dimensional substrate with a constraint on neighboring interface heights are described. The interface width obeys scaling even for small systems and grows as t^{β} with $\beta = 1/(d+1)$. Generalizations to include irrelevant effects such as noise reduction are discussed as are possible reasons for the discrepancies in earlier results.

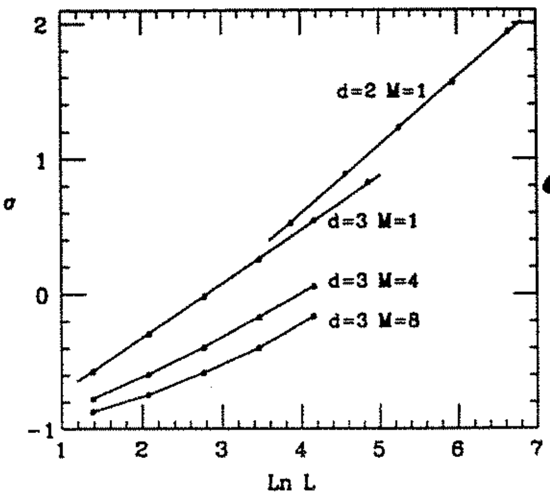


FIG. 1. Interface width σ as a function of system size L. M is the noise reduction parameter.

SATURATION WIDTH, ROUGHNESS EXPONENT

$$\chi \approx \begin{cases} 1/2 & d=2 \\ 2/5 & d=3 \end{cases}$$

"KK CONJECTURE"

$\Downarrow$

EARLY TIME EXPONENT

$$\beta = \frac{1}{d+1}$$

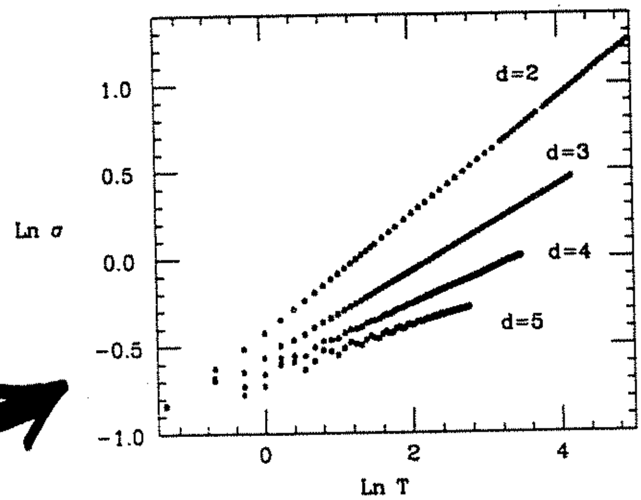


FIG. 2. Raw data of $\sigma(t)$ for the largest systems. Note the finite-size effects especially for $d=5$ $(L=32)$.

* EVIDENCE FOR SCALING ANSATZ

$$\sigma \sim L^{\chi} f(t/L^{z})$$

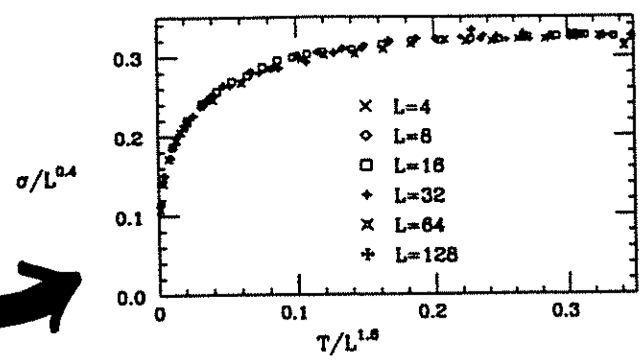


FIG. 4. Data collapse for $\sigma(L,t)$ in $d=3$ with $z=1.6$, $\chi=0.4$.

NOTE THAT SCALING RELATION $\chi + z = 2$ IS SATISFIED!

Disorder-induced roughening of diverse manifolds

Timothy Halpin-Healy
Department of Physics, University of Maryland, College Park, Maryland 20742
*and Department of Physics, Barnard College, Columbia University, New York, New York, 10027-6598**
(Received 15 January 1990)

We propose a unified treatment of the roughening of manifolds by impurities in quenched correlated random media. Our perspective incorporates such apparently distinct problems as domain walls in dirty Ising magnets, biased walks upon random lattices, and flux creep in high-T_c materials. By means of generalized Imry-Ma arguments and a functional renormalization group (RG), we find new results, including the random-bond interfacial roughening exponent, $\zeta_{RB}=2\epsilon/9$, as well as estimates for the many-dimensional directed-polymer wandering index. This last quantity is also investigated via real-space RG methods, where we find, for example, $\zeta_{2+1}=0.602$, in reasonable agreement with the functional RG value $\frac{3}{5}$. Finally, since the Burgers equation permits translation of our directed-polymer results to the Eden cluster and ballistic deposition problems in higher dimensions, we can compare to the most recent computer simulations of these stochastic growth models. In particular, we address issues regarding the exponent conjectures that have been made and suggest the possibility of a *finite* upper critical dimension.

I. INTRODUCTION

Ill-condensed matter,[1] both in its experimental manifestation as well as its realization via simplified models, such as random-field and random-bond Ising systems, directed polymers in disordered media, spin glasses, Eden clusters, ballistic deposits, and the like, has held the attention of the physics community for much of the past decade, remaining quite elusive and yielding but occasionally to our analytic investigative tools. Consider, for example, an Ising ferromagnet subject to quenched disorder in the form of random fields at each site.[2-12] Seemingly innocuous issues, such as those concerning marginal dimensionalities, critical exponents, and growth kinetics, proved to be sufficiently subtle that years of spirited, and sometimes contentious, discussion were required to resolve the controversies that arose. The random-bond problem[13-24] appears, presently, to be no less intractable. Nevertheless, it was realized rather early that many of the crucial static and dynamic bulk properties of these disordered magnets could be understood by examining the *impurity-induced roughening of domain walls* in these systems. The original insight, due to Imry and Ma,[2] has been generalized recently by Kardar[21] and Nattermann[22] to randomness of arbitrary correlation, thereby interpolating between random-field (RF) and random-bond (RB) situations. Halpin-Healy,[23] using a functional renormalization group (FRG), has subsequently retrieved all of the exact results of these authors for one-dimensional interfaces, but furthermore addressed the fundamental question of interfacial roughening in arbitrary dimensionality. Moreover, his treatment broadened the essentials in yet another respect—consideration of a vector, rather than a scalar, distortion field permitted unification of an impressive variety of problems, including domain-wall roughening,[2-22] directed-polymer wandering,[25-34] and flux creep and collective pinning in high-T_c superconductors,[35-38] all being subsumed under the heading of diverse manifolds in random media.[23] In addition, because of an explicit mapping,[39] via the Burgers equation,[40] connecting the directed-polymer problem to a host of others, such as the growth dynamics of Eden surfaces and ballistic deposites,[41-51] the large-time behavior of randomly stirred fluids,[52] and the asymptotics of driven diffusion,[53] there is an extraordinary richness to this subject.

The crucial object of interest in our investigations is the critical index $\zeta_{d',n}$, defined by the relation $|\mathbf{z}|\approx|\mathbf{x}|^{\zeta}$, which documents the disorder-induced roughening of an n-component vector field $\mathbf{z}(\mathbf{x})$ of d'-dimensional support. The central result following from our functional renormalization group, aside from Imry-Ma arguments generalized for diverse manifolds, is that *for the case of quenched, short-range correlated random media, the roughening exponent* $\zeta_{d',n}=2(4-d')/(8+n)$. The paper proceeds through an examination of several physically relevant instances, amongst them interfacial roughening (d' arbitrary, $n=1$, see Sec. II), directed-polymer wandering ($d'=1$, n arbitrary; Sec. III), as well as Eden clusters and ballistic deposits (Sec. IV). We also discuss briefly some recent work on collective pinning of vortex lines and flux creep in ceramic superconductors ($d'=3$, $n=2$; Sec. V), before closing with a summary (Sec. VI) of the salient features we hoped to communicate.

II. IMPURITY-STRICKEN MAGNETS

In this section, we will concentrate specifically on the subject of disorder-induced roughening of domain walls using the language of Ising magnets,[2-23] although our results have relevance in a variety of other contexts, including the pinning of charge density waves by impurities and interfacial delocalization in random media. As mentioned above, we consider a continuum model of the

13

domain wall embedded in a disordered environment. As always, we assume that the interface runs on average parallel to some reference plane with the single-valued function $z(\mathbf{x})$ denoting the position of the domain wall measured with respect to this plane at the point specified by the $(d-1)$-dimensional vector $\mathbf{x}$ (see Fig. 1). At length scales large compared to all microscopic features, the domain wall is properly described by the interfacial Hamiltonian

$$\mathcal{H}=\int d^{d-1}\mathbf{x}\left[\frac{\nu}{2}(\nabla z)^2+\sigma\mathcal{V}(z)\right],$$

where ν is the surface tension parameter that discourages deviations from planar and σ gauges the strength of the random pinning potential $\mathcal{V}$. It is precisely the competition between these two terms, elastic energy cost versus random energy gain, that determines the scaling properties of the roughened interface. We deal with correlated randomness that is Gaussian with zero mean and variance,

$$\langle \mathcal{V}(z,\mathbf{x})\mathcal{V}(0,0)\rangle=\delta^{d'}(\mathbf{x})R(z),$$

where $d'=d-1$ for notational convenience. Within the present formulation, the RB problem corresponds to impurities that are, strictly speaking, uncorrelated [i.e., $R(z)=\delta(z)$], while the RF Ising model involves correlations that increase linearly at large distances. The great virtue of this approach, inspired by the earlier work of Kardar[21] and Nattermann,[22] is that it handles domain-wall roughening by randomness of arbitrary correlation, thereby interpolating between the fairly well understood RF and apparently much more difficult RB problems. For functions R that are smooth at small argument and behaving asymptotically as $z^{-\beta}$, a simple scaling argument in the spirit of Flory, which in fact just generalizes the notions of Imry and Ma,[2] permits us to determine the roughening exponent ζ describing nonplanar fluctuations of the domain wall $z\sim|\mathbf{x}|^{\zeta}$. In particular, if we rescale lengths in the base plane according to $\mathbf{x}\to\lambda\mathbf{x}$, then $z\to\lambda^{\zeta}z$ and, furthermore, it is clear that ν and σ have scaling indices $y_\nu=2\zeta+d'-2$ and $y_\sigma=(d'-\beta\zeta)/2$, because the assumed variance form must remain unaltered, which requires $\mathcal{V}\to\lambda^{-(d'+\beta\zeta)/2}\mathcal{V}$. Insisting, as Imry and Ma did, that the surface tension cost balance the random energy gain (i.e., the two terms of the interfacial Hamiltonian must scale the same way) we fix the roughening exponent at $\zeta_F=\epsilon/(4+\beta)$, where $\epsilon=5-d$. Observe that this analysis retrieves, for $\beta=-1$, the known value $\zeta_{\rm RF}=\epsilon/3$, due to the original Imry-Ma argument, but sheds little light on the RB problem since it is not at all apparent which value of β is appropriate. Naively, we could suggest $\beta=1$, since the δ function has dimensions of inverse length; on the other hand, the δ function is shorter-ranged than any power law, so perhaps $\beta=\infty$ is most natural. Resolution of this ambiguity necessitates going beyond simple energetic issues associated with the trivial scaling properties of $\mathcal{H}$ as in the Imry-Ma scheme and examining fluctuations about this mean-field theory. Nevertheless, we suspect that provided the correlations are sufficiently long ranged (i.e., β small enough), the refined mean-field treatment will be correct. The crucial question then becomes—what is the critical β_c at which the theory breaks down and, most importantly, what is the roughening exponent $\zeta_{\rm SR}$ dictated by the new short-ranged fixed point function (SR FPF)? Application of the functional renormalization group readily provides some answers to these questions.

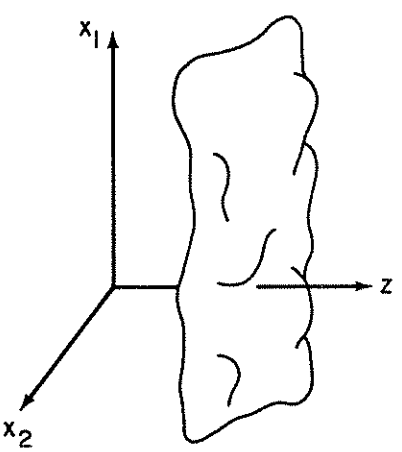


FIG. 1. A domain wall separating two phases of matter in d dimensions can be described conveniently by a single-valued function $z(\mathbf{x})$, denoting its height in terms of the remaining $(d-1)$ coordinates. Illustrated here is the most compelling case physically, $d=3$. In a pure system at zero temperature, such an interface would be strictly flat to minimize surface tension costs. However, if the medium were populated by quenched pinning impurities that attracted the domain wall locally, a competition would result between the random potential gain, favoring fluctuations, and the elastic energy cost, discouraging deviations from planar. What follows from this competition is a nontrivial disorder-induced roughening of the interface, embodied in the relation $z\sim|\mathbf{x}|^{\zeta}$, which characterizes typical transverse fluctuations of the domain wall with basal length. Such impurity-dominated roughening is usually much more severe than the mild thermal roughening that occurs at $T>0$ in the absence of disorder. Our interest lies in determining how the interfacial roughening exponent depends on the correlated nature of the quenched randomness.

Since we are interested in studying the statistical mechanics of domain walls in magnets subject to random, but correlated disorder, it is necessary to go beyond the tree-level Hamiltonian and to consider issues of free energy. Hence, we are led to an investigation of the full partition function:[18(a),23]

$$Z=\int \mathcal{D}z\,\exp\left[-\frac{1}{T}\int d^{d'}\mathbf{x}\left[\tfrac{1}{2}(\nabla z)^2+\mathcal{V}(z)\right]\right],$$

where T is the temperature; also, we are using units in which the surface tension ν is unity and the pinning potential strength is no longer explicit, but rather subsumed by the correlator R. Because R is completely general at this point and the functional RG will entail a thorough examination of the flow and renormalization properties of this function, we are completely free to avail ourselves of this convenience. As noted by Brézin and Orland,[19] the essence of the functional RG is merely a calculation of the effective action (or free energy) to one-loop order, followed by differential length $x \to x/(1+\delta l)$ and field $z \to z/(1+\zeta\delta l)$ rescalings, to determine the manner in which the interaction, here $R(z)$, behaves under the infinitesimal dilatation. Indeed, R is the relevant quantity in the present context because we desire to average the free energy $\ln Z$ over the many different realizations of disorder and must therefore rely upon the replica trick, which introduces R as the attractive interaction amongst replicas, following the (Gaussian integral) disorder average over $\mathcal{V}$. More explicitly, with angular brackets denoting as before the average over complexions of randomness, we invoke replicas to provide us with the apparently innocuous relation that connects the object we wish to know $\langle \ln Z \rangle$ with those we can actually calculate, namely, $\langle Z^N \rangle$:

$$\langle \ln Z \rangle = \left\langle \lim_{N\to 0} \frac{e^{N\ln Z}-1}{N} \right\rangle = \lim_{N\to 0} \frac{\langle Z^N\rangle - 1}{N} ,$$

there being little reason to believe the interchange of limiting and averaging procedures to be suspect in the case at hand (i.e., no replica symmetry breaking). In this fashion, we focus our attention on the disorder-averaged, replicated interface partition function,

$$\langle Z^N \rangle = \int \mathcal{D}z_\alpha \exp\left[\frac{-1}{2T} \int d^{d'}x \times \left[\sum_\alpha (\nabla z_\alpha)^2 - \frac{1}{T} \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} R(z_\alpha - z_\beta) \right] \right]$$

the sum over $\alpha,\beta = 1,\ldots,N$. Formally expanding the correlator R about its minimum to quadratic order, going to Fourier space and evaluating the resulting Gaussian integral in the standard manner yields the one-loop effective action:

$$-\frac{R_{\rm eff}}{T^2} = \tfrac{1}{2}\mathrm{Tr}\ln\left[\delta_{\alpha\beta} + \frac{1}{q^2T} R''(z_\alpha - z_\beta) - \frac{\delta_{\alpha\beta}}{q^2T} \sum_\gamma R''(z_\alpha - z_\gamma) \right] ,$$

where $\mathrm{Tr} = \sum \int d^{d'}q$. The calculation proceeds by using the matrix analog of the Taylor-series expansion $\log(1+x) = x - x^2/2 + \cdots$ and concentrating on the second-order piece, which gives us the desired factor of T^{-2} to balance that on the left-hand side. This is necessary because our functional RG, while treating the flow properties of R, is essentially *perturbative in the temperature.* In fact, a moment's reflection upon the full partition function reveals that T has length dimensions $[T] = d' - 2 + 2\zeta$, so that under the infinitesimal rescaling we have $T \to T_R = T/(1+\delta l)^{d'-2+2\zeta}$, which implies that the differential flow equation for the temperature reads

$$\frac{dT}{dl} = (2 - d' - 2\zeta)T .$$

Hence, provided the disorder-induced roughening is more severe than simple thermal roughening $[\zeta > (2 - d')/2]$, $T \to 0$ under renormalization and the physics is controlled by a *zero-temperature fixed point.* Indeed, this is typically the case for interfaces in the physically relevant dimensions and, in addition, has the very important effect of allowing us to ignore the linear term in our expansion of the one-loop logarithm, since there is an uncompensated power of T. By contrast, the cubic and higher-order pieces can be neglected because, as we will show shortly, the SR FPF is a Gaussian-damped power law at large argument and our interest concerns only the tail of this function. In sum, the surviving term at quadratic order is the leading and sole relevant one. It reads

$$\tfrac{1}{2}(-\tfrac{1}{2}) \left[\sum_{\alpha,\beta} [R''(z_\alpha - z_\beta)]^2 - 2\sum_{\alpha,\beta,\gamma} R''(z_\alpha - z_\beta)\delta_{\alpha\beta} R''(z_\beta - z_\gamma) + \sum_{\alpha,\beta,\gamma} R''(z_\alpha - z_\beta) R''(z_\alpha - z_\gamma) \right] \int_0^\Lambda \frac{d^{d'}q}{q^4T^2} .$$

As we are performing an infinitesimal dilatation, the fluctuation correction to the effective action involves a change:

$$-\frac{1}{2T^2}\left[\sum_{\alpha,\beta} [R''(z_\alpha - z_\beta)]^2 - 2\sum_{\alpha,\beta} R''(z_\alpha - z_\beta) R''(0) \right] \delta l ,$$

where, for convenience, we have set the momentum cutoff $\Lambda = 1$ and ignored the term involving an irrelevant three replica sum.[18(b)] Of course, when we include the trivial rescaling

$$R \to [1 + (\epsilon - 4\zeta)\delta l] R(z[x(1+\zeta\delta l)]$$
$$\to R + [(\epsilon - 4\zeta)R + \zeta z R']\delta l ,$$

following from the simple length dimensions of R and z, we arrive at the functional recursion relation for the corrlator,

$$\frac{\partial R}{\partial l} = (\epsilon - 4\zeta)R + \zeta z R' + \tfrac{1}{2}(R'')^2 - R''R''(0) ,$$

a nonlinear partial differential equation first written down, though derived in a slightly different manner, by Fisher.[18(a)]

Unfortunately, when Fisher first obtained this partial differential equation, the ideas of Kardar[21] and Nattermann[22] were not yet available and there was little to be done, save for a straightforward numerical integration to search explicitly for the scale-invariant ($\partial_l R^*_{\rm SR} = 0$) SR FPF relevant to the RB problem. Intuitively, we would expect $R^*_{\rm SR}$ to be a Gaussian of sorts—though the only physically motivated analytic properties we insist upon are (i) short-ranged character resulting from renormalization of the bare δ function, (ii) zero slope at the origin, since R is assumed to be even, and finally, (iii) an asymptotic approach to zero at large argument. In fact, imposition of this last requirement permitted Fisher to ascertain the RB roughening exponent $\zeta_{\rm SR}$ as the numerical solution to an eigenvalue problem. The computational method is as follows. Consider the equation satisfied by the SR FPF. Rescale R by ϵ and then set $R''(0)=-1$. We are at liberty to do this, since it amounts to choosing, separately, our units of length along the R and z axes, different rulers permitted. The choice, of course, is arbitrary in that it will have no effect whatsoever upon the final answers, although is convenient for our purposes, since it has an ordering effect. The resulting nonlinear differential equation reads

$$(\epsilon-4\zeta)R+\zeta zR'+\tfrac{1}{2}(R'')^2+\epsilon R''=0 .$$

Our numerical integration must commence at the origin. Since the first and second derivatives there have already been fixed, the sole degree of freedom lies in picking $r=R(0)$. In fact, a choice for r amounts to guessing the eigenvalue ζ, as can be seen by evaluating the above equation at $z=0$:

$$\zeta=\frac{\epsilon}{4}\left[1-\frac{1}{2r}\right] .$$

For $r=2.997$, which implies $\zeta\approx0.2083\epsilon$, we retrieve Fisher's estimate and a SR FPF that approaches zero asymptotically (Fig. 2). Moreover, as pointed out by Fisher, the functional form of the tail can be determined exactly, $R\sim z^{-5+\epsilon/\zeta}\exp(-z^2/2\epsilon)$. The importance of this will be made quite clear shortly. For larger values of r, the numerical integration yields a function that goes negative at some finite value of z, bottoms out and then approaches the axis from the underside. By contrast, if r is chosen too small, the curve appears to level out at a value that is certainly nonzero. Naturally, as regards the SR FPF, both of these behaviors are unacceptable. In practice, we decreased r until the intercept moved off to infinity—the result is robust to the four quoted significant digits. While there may be no reason to object to this particular implementation of the numerical integration, we[23] have recently emphasized that the starting point may be problematic. That is Fisher arrived at his differential equation for the SR FPF by neglecting the higher-order terms following expansion of the one-loop logarithm. Although such terms may be ignored when discussing the large-z behavior, they surely cannot for $z=0$. Yet the numerical integration procedure commences precisely at this point. In short, the differential equation is correct asymptotically, though only approximate near the origin; nevertheless, we will see that Fisher's estimate is surprisingly good.

Aided by the hindsight furnished by Karder and Nattermann, we sought to view these issues within the broader context of models interpolating between RF and RB problems. More specifically, we realized that the only exact information obtainable from Fisher's differential equation was the functional form of the tail, as mentioned above. But therein lay the answer, since the SR FPF represented a Gaussian damping of the critical power law $z^{-\beta_c}$. Hence, we have a second equation $\beta_c=5-\epsilon/\zeta$ to complement the generalized Imry-Ma formula $\zeta=\epsilon/(4+\beta)$ which we know to be valid up to and including the critical β. Simultaneous solution of these two equations yields

$$\zeta_{\rm SR}=\tfrac{2}{9}\epsilon,\quad \beta_c=\tfrac{1}{2} .$$

For $\beta>\beta_c$, the roughening exponent ζ remains at this value and no longer obeys the Flory formula. In Fig. 3, we summarize our findings for the interfacial roughening index for randomness of arbitrary correlation. The interesting new prediction is that $\zeta_{\rm RB}=2\epsilon/9$, compared to the Imry-Ma determined $\zeta_{\rm RF}=\epsilon/3$. As suspected, inter-

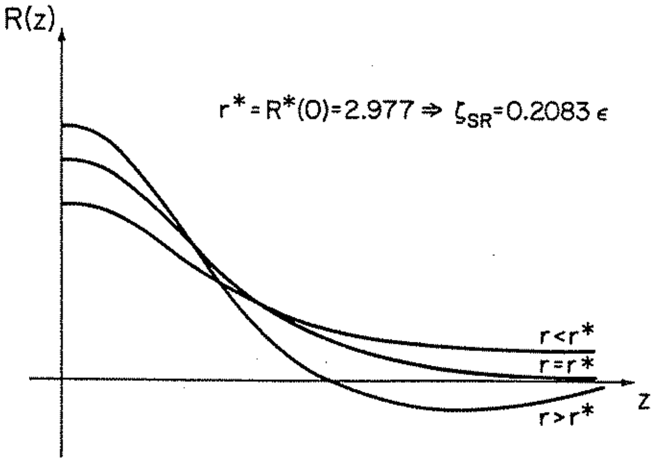


FIG. 2. Numerical analysis of Fisher's second order, nonlinear PDE for the SR FPF, $R^*(z)$, controlling interfacial roughening by quenched randomness of short-ranged correlation. The integration commences at the origin, where the slope vanishes by symmetry and we have set $R''=-1$ for convenience. Our sole remaining freedom is in choosing $r=R^*(0)$, which being related to ζ, thereby fixes the roughening exponent as a numerical eigenvalue. For $r=r^*=2.997\Longrightarrow\zeta\approx0.2083\epsilon$, we find a numerically determined FPF with the correct asymptotic behavior. By contrast, picking $r>r^*$ yields a correlator that goes negative, while $r<r^*$ gives a falloff that does not approach zero.

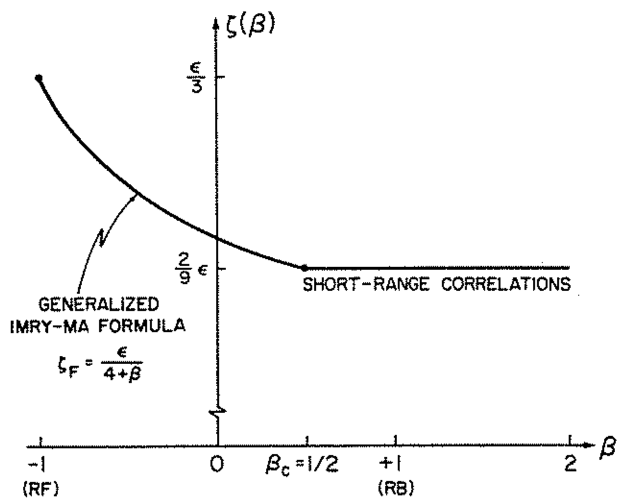


FIG. 3. Interfacial roughening exponent ζ as a function of β, which describes the decay of impurity correlations. A generalized Imry-Ma theory is correct only for sufficiently long-ranged correlations. Beyond the critical value $\beta_c=\frac{1}{2}$, a single SR FPF dictates the scaling properties and ζ sticks to the RB value $2\epsilon/9$.

(16)

faces are roughened considerably more by RF's than they are by RB's. *Correlated disorder incurs greater wandering.* Lastly, observe that since $\beta_c < 1$, the ambiguity that arose earlier regarding the bare R most appropriate to the RB case is now rendered moot, because all functions falling off faster than the critical power law, once renormalized, lead to the same asymptotic scaling properties. This is a manifestation of the functional formulation of the renormalization group.

Knowledge of these wandering exponents permits us to make a number of additional interesting physical predictions. With regard to the domain-wall problem, it is known that for impurity-stricken Ising systems quenched into the ferromagnetic phase, ζ fixes the growth rate of ordered, coherent regions.[10,13] In fact, at long times, the characteristic domain sizes scales as $(\ln t)^{1/\Psi}$, where $\Psi=(d-3+2\zeta)/(2-\zeta)$. Consequently, for RB disorder, we obtain the new result $\Psi^{\mathrm{RB}}(d=3)=\frac{4}{7}$. In addition, $\Psi^{\mathrm{RB}} \approx 1-\epsilon/6$ as one approaches dimension five. Another application involves the phenomenon of interfacial wetting in random systems.[16] For complete wetting in the weak fluctuation regime, the mean position of the interface is determined by the competition between the external field, which pushes the interface towards the wall, and the delocalizing effects of the randomness. As the bulk field vanishes, the interface is depinned by the impurities, its mean distance from the wall diverging with an exponent $\nu_\perp=\zeta/(2-\zeta)$. Therefore, in $d=3$ with uncorrelated quenched disorder where $\zeta_{\mathrm{RB}}=\frac{4}{9}$, we expect the complete wetting index $\nu_\perp=\frac{2}{7}$.

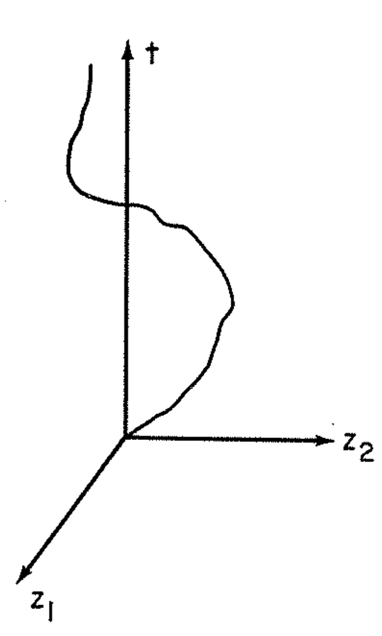


FIG. 4. Impurity-induced roughening of an elastic linear defect, such as a vortex line in a dirty ceramic superconductor, dislocation line in a disordered solid, or a directed polymer in a random medium. The transverse position of the elastic string is given by an n-component vector field $\mathbf{z}(t)$, time t suggestively labeling the special axis of anisotropy. Shown explicitly here is the case $n=2$. The wandering exponent ζ, defined via the relation $|\mathbf{z}|\sim t^{\zeta}$, describes the scaling of this object which results from the competing effects of elasticity and the uncorrelated random pinning potential.

III. DIRECTED POLYMERS

A. Functional renormalization group

As the above application of the functional RG appears fairly powerful, it behooves us to generalize our results for the scalar field, which is appropriate to the discussion of interfaces, to the case of an n-component vector field $\mathbf{z}$. In addition, if for the moment, we focus our attention upon the situation where the base space has but a single dimension (calling this special direction time t) and restrict ourselves to uncorrelated disorder, we are led rather naturally to a Euclidean path integral lending itself to a very attractive physical interpretation, namely, the space-time diagram of a point particle subject to a dynamically random potential (see Fig. 4). Alternatively, we could view this partition function as describing any linear defect in a quenched random environment, whether it be a vortex line in an impure high-T_c superconductor,[36] a dislocation in a disordered solid,[20] or a directed polymer (a polyelectrolyte, perhaps) in a gel matrix.[25] Regardless of the chosen interpretation, however, the essential physics remains the same and concerns the optimization of a biased walk amidst impurities. Anisotropy (i.e., the existence of the preferred time direction) and randomness are the crucial aspects of this problem and, indeed, the fundamental quantity of interest is the so-called wandering (or, as before, roughening) exponent ζ, which describes how transverse fluctuations scale with longitudinal length $|\mathbf{z}|\sim t^{\zeta}$ after we perform the statistical averages over many realizations of the disorder. Beyond ζ, there is also the index ω, which tracks the fluctuations in the energy of the optimal path as we consider various configurations of randomness. Nonetheless, an exponent relation[13,14] $\omega=2\zeta-1$ following from the simple scaling properties of the kinetic piece of the hamiltonian, connects the two.

The above directed-polymer problem (we will, for convenience, henceforth discuss its various formulations under this single heading) has proved to be quite formidable despite the utter simplicity with which it is posed. In particular, all attempts to determine explicitly how the wandering exponent ζ depends on the number of transverse dimensions n have thus far appeared uniformly unsuccessful. Moreover, it is not even clear if directed polymers in random media possess a finite upper critical dimension (UCD), beyond which the roughening exponent retrieves its trivial thermal (or entropic) value $\zeta_{\mathrm{th}}=\frac{1}{2}$. To date, what is known with certainty is that for the case $n=1$, where the directed polymer (DP) behaves as the one-dimensional interface of a 2D RB Ising magnet $\zeta_{1+1}=\frac{2}{3}$, an exact result due to several independent investigators.[14,15] In addition, Derrida and Spohn[29] have addressed the problem on a Cayley tree, finding $\omega=0$ (logarithmic behavior), which suggests that at least in infinite dimensionality, wandering may be simply entropic. Unfortunately, these are the only exact analyses. Various computer simulations[47–49] on stochastic growth models, such as ballistic deposition and Eden clusters, when translated into the language of directed polymers (see later) reveal a decreasing many-dimensional DP index. Specifically, the Wolf and Kertész (WK) conjecture requires $\zeta_{WK}=(n+1)/(2n+1)$, while that of Kim and Kosterlitz (KK) has $\zeta_{KK}=(n+3)/(2n+4)$. Both formulas recover the expected behavior in infinite dimensionality and assume no finite UCD. In this section, we will discuss briefly the approximate functional RG calculation[23–24] of the wandering exponent ζ, while Sec. III B will describe our efforts[34] using a real-space renormalization group (RSRG).

Generalizing our earlier considerations (see Sec. II) to the case of an n-component vector field yields some additional terms on the right-hand side of our differential flow equation. They are of the form[54]

$$\frac{n-1}{2}\left[\frac{R'}{z}\right]^2-(n-1)\frac{R'}{z}R''(0)$$

and follow immediately from the fact that

$$R_{ij}=\left[\delta_{ij}-\frac{z_iz_j}{z^2}\right]\frac{R'}{z}+\frac{z_iz_j}{z^2}R'',$$

where Latin indices denote partial derivatives with respect to the components of $\mathbf{z}$ and we have made the simplest, but physically motivated, assumption that the correlator R is only a function of the magnitude $z=|\mathbf{z}|$. (However, Nattermann,[24] has shown that the same flow equation follows regardless of this choice.) With these terms in hand, we can proceed to determine the analytic behavior of the tail of our new SR FPF, just as we did for the interfacial problem ($n=1$). We discover a Gaussian-damped critical power law with $n\beta_c=4+n-\epsilon/\zeta$. Supplementing this equation by the generalized Imry-Ma result $\zeta=\epsilon/(4+\beta n)$ which we know to be correct up to and including β_c, we solve and find that $\zeta_{SR}=2\epsilon/(8+n)$ and $\beta_c=\frac{1}{2}$, the fundamental FRG result of this paper. Evidently, within our scheme, the value of the critical falloff is superuniversal (i.e., independent of both n and d). Specializing to the DP problem, where the base space is unidimensional ($\epsilon=3$), we have the following formula for the many-dimensional directed polymer index:

$$\zeta_{DP}=\frac{6}{8+n}.$$

But this result is not to be trusted for $n>4$, since an examination of the thermal flow equation, when rewritten for DP's,

$$\frac{dT}{dl}=(\tfrac{1}{2}-\zeta_{DP})T\ ,$$

reminds us that the zero-temperature fixed point loses stability there, whereas our FRG relies inherently upon a renormalized $T\to 0$. Nevertheless, assumption of the *most naive scenario* would have the temperature run off to infinity in this case (if there are no intervening FP's), thereby fixing $\zeta=\frac{1}{2}$ for all $n\geq 4$ (see Fig. 5). It is in this manner that we propose a mechanism for a *finite UCD to the DP problem*.[50] Yet the behavior of directed polymers at nonzero temperatures raises subtle issues, as pointed out by Nattermann. If we employ a Harris criterion

$$\alpha=2-\nu_{\parallel}d_{\parallel}-\nu_{\perp}d_{\perp}=2-1-\tfrac{1}{2}n=1-\tfrac{1}{2}n>0\ ,$$

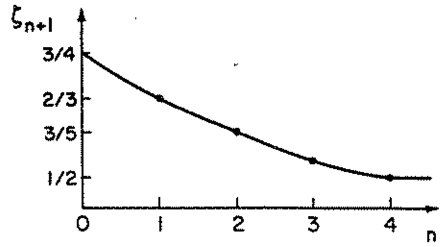


FIG. 5. Wandering exponent ζ for the many-dimensional directed polymer. Given the most naive assumption regarding renormalization-group flows, the roughening is simply entropic ($\zeta=\zeta_{th}=\frac{1}{2}$) for $n\geq 4$.

we find that the infinite temperature (purely entropic wandering) fixed point is itself unstable to disorder for $n\leq 2$. Hence, within the context of our FRG approach, the renormalization flows we find in temperature space are summarized in Fig. 6. Note the existence of a finite-temperature phase transition for $2<n<4$, separating disorder-induced and thermal roughening.[51] The marginality of $n=2$, and the strong-coupling nature of the problem were painfully apparent in early investigations of directed polymers that relied upon the Burgers's equation, an approach which has the misfortune of being perturbative in the disorder. By contrast, the FRG, since it is perturbative in the temperature rather than the disorder, can gain access, albeit approximate, to the nontrivial fixed point of great interest. However, it eventually loses validity for $n>4$. Hence, there the figure should be viewed as speculative. Nonetheless, the possibility of a finite UCD to the directed polymer problem is an intriguing notion very much worthy of further attention.

B. Real-space renormalization group

Given the rather severe difficulties encountered in determining exactly the many-dimensional directed-polymer wandering exponent, one is forced to rely upon

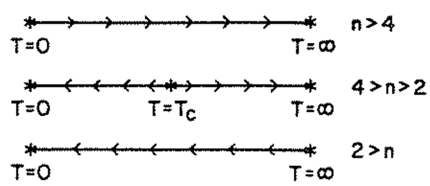


FIG. 6. Renormalization group flows for the temperature T for directed polymers in n transverse dimensions. For $n>4$, the functional RG approach breaks down, losing consistency since the temperature flows from $T=0$. Given that the thermal fixed point is perturbatively unstable to disorder, we illustrate the simplest possible global scenario.

the trusted panoply of approximate techniques to make any progress. Beyond the FRG,[23,24] these nonperturbative tools include transfer matrices, the real-space renormalization group, as well as potential insights via instantons. Indeed, it was precisely the transfer-matrix work of Kardar and Zhang[25] which brought to light the richness of the subject of directed polymers in random media. Although their simulation first suggested superuniversal exponents, subsequent investigations revealed that the indices ω and ζ decreased slightly with dimensionality. In this section, we discuss attempts via a RSRG introduced by Derrida and Griffiths[30] that was recently extended[34] by the present author to document the gradual decline of these critical exponents towards their mean-field values.

The Derrida-Griffiths RSRG, a variant of the Migdal-Kadanoff method specifically adapted to situations with quenched random disorder, is a procedure that, at least in principle, is exact upon the so-called hierarchical lattices—those carefully crafted lattices whose connectivity is such that bond moving incurs no approximation (see Fig. 7). If we consider the ith generation of the b-branch hierarchical lattice illustrated in the figure, there are b^{i-1} possible directed paths between the points A and C, each of which has length $L=2^{i-1}$. In the directed polymer problem on disordered hierarchical lattices, we associate random energies with each of the bonds and search for the optimal paths of least energy. Averaging over many realizations of randomness and studying the statistical properties of the optimal path's energy as a function of the trajectory length L, we can ascertain the energy fluctuation exponent ω, since rms deviations from the mean will scale as L^{ω}. Within the context of the Derrida-Griffiths RSRG, the fundamental object of interest is the probability distribution $P(x)$, from which the uncorrelated random-bond energies are drawn. The crucial feature of the method, however, is a recursion relation which dictates the manner in which this probability distribution is transformed after successive generations as one is looking, effectively, at larger and larger length scales. With P_i denoting the functional form of the probability distribution at the ith generation, the renormalized distribution at the next generation is obtained via a multistep process. Firstly, splitting each bond in two (i.e., doubling the path length $L \to 2L$) necessitates a self convolution of the probability distribution

$$Q_i(x)=\int dy\, P_i(x-y)P_i(y)\ ,$$

while the b-fold replication of the branch requires

$$\int_x^{\infty} P_{i+1}(y)dy=\left[\int_x^{\infty} Q_i(y)dy\right]^b$$

so that the $(i+1)$th generation probability distribution is given by

$$P_{i+1}(x)=bQ_i(x)\left[\int_x^{\infty} Q_i(y)dy\right]^{b-1}\ ,$$

which, in fact, is the form of the recursion utilized most naturally in practice. The appropriateness of the second step was made particularly manifest by the observation of Derrida and Griffiths[30] that in order for the optimal path's energy not to exceed a certain value, it must not do so in each of the newly created b branches. Hence, it is the bth power of the partially integrated convolution that is relevant and must be equated to a similar integration of the renormalized distribution. For a given brand of hierarchical lattice (that is, for a specific value of b), we determine the energy fluctuation exponent ω following from the Derrida-Griffiths RSRG by iterating the recursion relation for the probability distribution P and noting the evolution of its width after successive generations. As the path length doubles from one generation to the next, the width of P ought to increase by a factor that asymptotically approaches a value $\lambda=2^{\omega}$.

Guided by a strong faith in the principle of universality and motivated by calculational convenience, we have restricted ourselves to initial (or, using the language of the renormalization group, *bare)* probability distributions of the form

$$P_1(x)=p\delta(x-1)+(1-p)\delta(x+1)\ ,$$

with $p=0.95$ sufficiently large, for the dimensionalities of interest, to guarantee that the (-1) bonds do not percolate across the lattice, while simulatneously maximizing smooth, quick convergence of the successive λ_i. Iteration of the recursion relation leads to an ever-growing collection of δ-function spikes whose coefficients and center of mass are altered with each new generation. The numerical aspects of this procedure are handled with relative ease on a computer. In Fig. 8, we illustrate our results for the particular case $b=4$, indicating the form of the renormalized probability distribution for generations $i=3,6,9,12,15$. The most transparent features are the decreasing height and increasing width of successive iterates (the RG, of course, preserves normalization) as the renormalized distribution runs off in the direction of $-\infty$. The accompanying inserts document the absolute width δ_i, in addition to the width renormalization factor $\lambda_i=\delta_i/\delta_{i-1}$, as a function of the generation i. After a dozen iterations, the RG procedure is well en route to asymptopia (certainly for b this small) and we attain

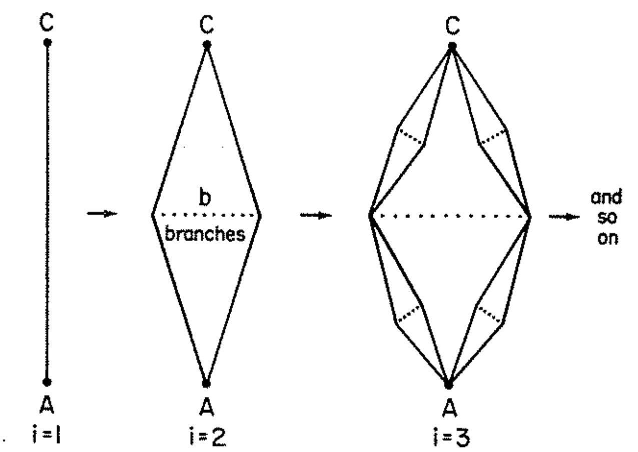


FIG. 7. b-branch generalized hierarchical lattice, connecting the points A and C, shown through the third generation ($i=3$). A succeeding generation is obtained from the previous one by splitting each bond in half, followed by a b-fold replication.

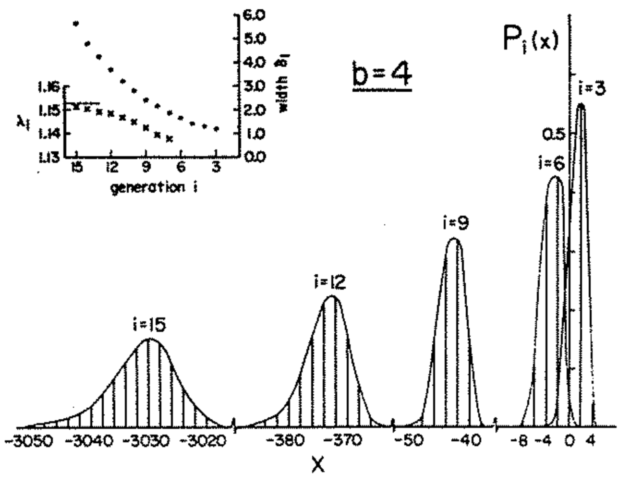


FIG. 8. Renormalized probability distributions $P_i(x)$ at generations $i=3,6,9,12,15$, using the Derrida-Griffiths RSRG technique for directed polymers on a $b=4$ disordered hierarchical lattice. The inset shows the absolute rms width δ_i of the iterated distribution, as well as the width renormalization factor $\lambda_i=\delta_i/\delta_{i-1}$, which asymptotically approaches the value $\lambda \approx 1.152$.

$\lambda=\lim\lambda_i \approx 1.152$, which implies an energy fluctuation exponent $\omega(b=4)=\ln_2\lambda \approx 0.204$, there being an uncertainty of ± 2 in the last digit associated with our terminating at the fifteenth iteration. In Fig. 9, we summarize our results concerning ω for $b \le 12$, using the Derrida-Griffiths RSRG. With the standard, but approximate, correspondence between b-branch hierarchical and d-dimensional hypercubic lattices $b=2^{d-1}$ we can connect immediately to the physically relevant cases, showing explicitly our estimates for the directed-polymer wandering exponent

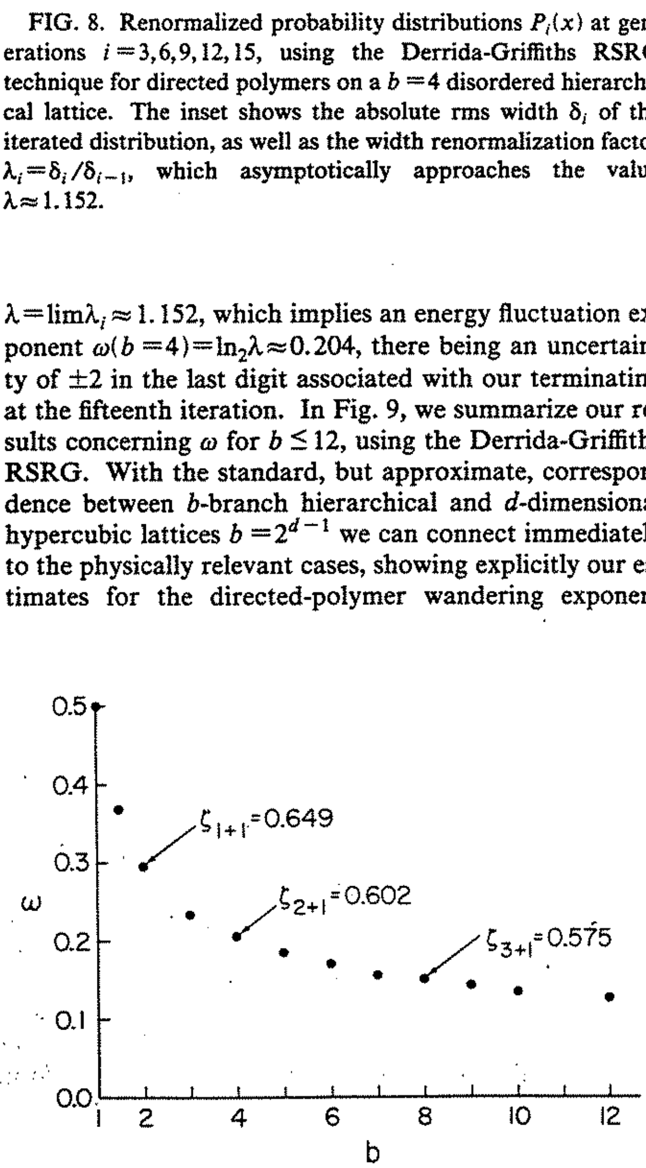


FIG. 9. Energy fluctuation exponent $\omega=\ln_2\lambda$ for general b, as obtained by real-space methods. Using the standard, but approximate, correspondence between b-branch hierarchical and d-dimensional hypercubic lattices, $b=2^{d-1}$, we can translate these results to the cases of greatest physical relevance. Indicated explicitly are our estimates for the many-dimensional directed polymer wandering exponent $\zeta=(1+\omega)/2$, for $d=n+1$.

$\zeta=(\omega+1)/2$. The general trend concerning the dimensionality dependence of the critical indices is clear. For $d=1$, the exact value $\omega=\frac{1}{2}$ is retrieved, followed by a steady decline toward zero as d is increased, lending much doubt to the previously advanced notion of stochastic superuniversality. As pointed out by Derrida and Griffiths,[30] the result for $d=2$ is a bit below the known wandering exponent $\zeta=\frac{2}{3}$. The great interest, of course, is for higher dimensionalities, which have thus far eluded exact analysis; whereas the functional RG yields $\zeta_{2+1}=\frac{3}{5}$, the RSRG value mentioned above requires 0.602, which is certainly consistent, though somewhat greater. We might add that it is especially so were the RSRG result yet another underestimate. Finally, the Derrida-Griffiths RSRG gives $\zeta_{3+1}\approx 0.575$, considerably larger than the approximate functional RG value. Unfortunately, for $b>12$, precise determination of ω is difficult, rendering unlikely any measure of the UCD. Nevertheless, the decline of the exponents does appear to be asymptotic. Indeed, if this is the case and the directed polymer, Eden cluster and ballistic deposition (see next section) problems do lack a finite UCD, a $1/d$ expansion for the critical exponents may bring some light to these issues. Oddly enough, however, preliminary investigations[32,33] in this direction suggest that the scaling may remain trivial within a neighborhood of infinite dimensionality.

IV. EDEN CLUSTERS AND BALLISTIC DEPOSITION

Thanks to the work of Kardar, Parisi, and Zhang (KPZ),[39] it is much appreciated these days that the apparently intractable directed polymer problem is rather closely related to a set of stochastic growth models, including ballistic deposition and Eden clusters,[41–51] which are correctly described by the nonlinear Burgers equation

$$\partial_t h=\nabla^2 h+\tfrac{1}{2}(\nabla h)^2+\eta(\mathbf{x},t)$$

supplemented by random, uncorrelated space and time-dependent noise η. Here, h is the local height of an Eden cluster that grows upon a bed of seed particles forming the $(d-1)$-dimensional substrate. Recall that for the Eden model, the mechanism of stochastic growth is random occupation, with equal probability, of any perimeter site, whereas in ballistic deposition, particles are dropped at random locations from above, follow a strictly vertical descent and then stick to the deposit on contact. Either way, iteration of the procedure leads to a compact, nonfractal object with a propagating surface whose roughness has interesting scaling properties. Indeed, the breaking of time-reversal invariance due to propagation in a particular direction gives rise to the nonlinear term above. This notion was suggested first by KPZ and then observed explicitly by Plischke, Rácz, and Liu[43] in a simulation with simultaneous adsorption and emission.

Implementation of the Eden growth rule on a computer using the so-called "strip geometry," see Fig. 10, where one considers a bin or container of transverse linear dimension L, the following behavior is typically noted for w, the rms surface roughness. At short times, the width increases as a power law $w \sim t^{\beta}$, with a charac-

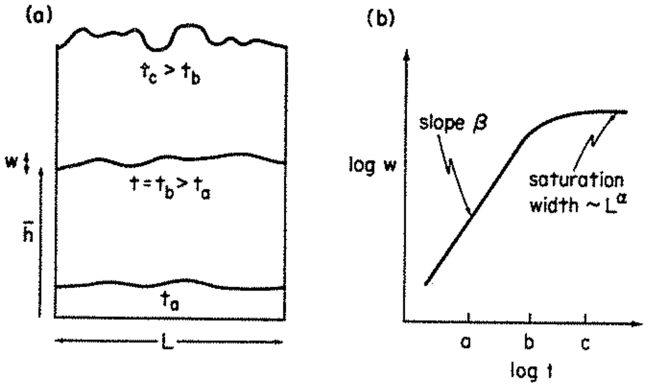


FIG. 10. (a) Growth of a d-dimensional Eden cluster or ballistic deposit in the strip geometry, proceeding vertically, shown at early (t_a), intermediate (t_b), and late (t_c) times. The bin has basal area L^{d-1}, while the propagating surface has mean height h and rms roughness w. (b) Logarithmic plot of surface width w vs time t typically observed in computer simulations. At short times, the roughness increases as a power law with exponent β, whereas at large times it saturates at some finite-size dependent value L^{α}.

teristic exponent β. There is a crossover phenomenon at intermediate times. Finally, much later, when the height greatly exceeds the base, the surface width saturates at some L-dependent value $w \sim L^{\alpha}$. A scaling ansatz consistent with this behavior has the form $w \sim L^{\alpha}\phi(t/L^{z})$, where for large argument the scaling function ϕ approaches a constant and for small argument it goes as a power law with the necessary exponent $\beta = \alpha/z$. Note that the dynamic exponent z, a notation drawn from the literature of critical phenomena, sets the time scale for the inevitable finite-size induced crossover. Nevertheless, it is known that this problem possesses only a solitary independent exponent, there being an index relation[39,42,44]

$$\alpha + z = 2$$

revealing that the sum of the static and dynamic exponents is always the same, regardless of dimensionality. Unfortunately, beyond this, the only existing exact results are for Eden clusters and ballistic deposits in $d=2$, where an analysis of Burgers's equation has yielded $z=\frac{3}{2}$.[15,39] The strong-coupling exponents in higher dimensions have remained stubbornly elusive, however, since for $d \geq 3$, the problem has a nonperturbative nature which renders the methods of KPZ essentially useless. Yet, there has been no shortage of computer simulations attempting to ascertain the growth exponents. The best of the previous generation were due to Wolf and Kertész on Eden clusters,[47] who found $\alpha(d=3)=0.33\pm0.01$ and $\alpha(d=4)=0.24\pm0.02$ for the long-time exponent which they could determine with the greatest precision. Since it is known that $\alpha(d=2)=\frac{1}{2}$, their results led them to conjecture that $\alpha=1/d$. More recently, Kim and Kosterlitz[48] simulated a solid-on-solid (SOS) model of ballistic deposition, arriving at a competing conjecture $\beta=1/(d+1)$, having discovered with great care that $\beta(d=2)=0.332\pm0.005$, $\beta(d=3)=0.250\pm0.005$, and $\beta(d=4)=0.20\pm0.01$. Of course, both of these conjectures, aided means of the scaling relation, reproduce exact results for $d=2$. Most importantly, though, the two also suggest implicitly that the Eden clusters and ballistic deposition problems lack a finite upper critical dimension. In addition, it might be pointed out that the KK conjecture yields the correct answer for the trivial case $d=1$ as well, although we may not be justified in attributing failure to WK in this very special dimension. Finally, Forrest and Tang[49] have just reported extraordinarily precise estimates of the short-time exponent in higher dimensions $\beta(d=3)=0.240\pm0.001$ and $\beta(d=4)=0.180\pm0.005$, which they manage via a mapping between SOS and Potts models on impressively large lattices. Their values are close, but distinctly lower than those of KK.

The relationship between the directed polymer and stochastic growth problems is made explicit via the nonlinear Burgers equation and the simple substitution $h(x,t)=\ln Z(x,t)$, which gives an imaginary-time Schrödinger equation

$$\partial_t Z = (\nabla^2 + \eta) Z$$

for a particle subject to a dynamically random pinning potential—such is the new role played by the uncorrelated stochastic noise η. Some thought reveals that the above non-Markovian diffusion equation is simply the differential formulation of the Feynman path integral (or partition function Z) introduced in Sec. III for the directed polymer in random media. Moreover, since we have identified the height h of the Eden cluster and ballistic deposit with the free energy $\ln Z$ of the directed polymer, it is clear that the fluctuations of these quantities about their respective means should have the same temporal scaling properties; that is,

$$\beta_{\text{Eden}} = \omega_{\text{DP}} \, .$$

Similar reasoning necessitates the identification of $1/z$ and ζ, although this falls out rather naturally from the scaling law.

The important point, of course, is that there really is only a single exponent in all this. Hence, the dynamic index $z=\frac{3}{2}$ of Eden clusters and ballistic deposits in $d=2$, is just the wandering exponent $\zeta=\frac{2}{3}$ for the $(1+1)$-dimensional directed polymer. Ignorance of the many-dimensional directed polymer index translates into lack of knowledge regarding Eden clusters and ballistic deposits in higher dimensions. Nevertheless, we can compare our functional and real-space RG estimates, obtained within the context of directed polymers, with the conjectures advanced in the realm of stochastic growth models. In $d=3$, we have

$$\beta_{\text{FRG}} = \tfrac{1}{5}, \quad \beta_{\text{RSRG}} = 0.204 \, ,$$

the first in explicit agreement with the WK value, the second an apparent confirmation, though noticeably greater. All, however, are clearly less than the KK prediction of $\frac{1}{4}$, well outside KK's quoted uncertainties. In the opinion of the author, this dimensionality, which after all is the most physically relevant, will be the proving ground upon which these and future conjectures may fall. For $d=4$, $\beta_{\rm FRG}=\frac{1}{11}$, $\beta_{\rm WK}=\frac{1}{7}$, $\beta_{\rm RSRG}=0.15$, $\beta_{\rm FT}=0.18$, $\beta_{\rm KK}=\frac{1}{5}$. If, as we suspect, the RSRG systematically underestimates the critical exponents, some doubt is cast upon the WK and functional RG conjectures. However, were the latter to hold true, a most interesting ramification would be the existence for the Eden cluster and ballistic deposition problems of a finite UCD $d_u=5$, beyond which the surface scaling properties are elementary. Needless to say, though provocative, this notion[50] is very much at odds with the prevailing wisdom as embodied in the ideas of WK and KK.

V. COLLECTIVE FLUX CREEP IN CERAMIC SUPERCONDUCTORS

Recently, investigators[35–38] of high-T_c superconductivity have concerned themselves with the physical properties of the Abrikosov flux line lattice (FLL) exhibited by these materials in an external magnetic field. In particular, there appears to be an unfortunate tendency towards giant thermal flux creep,[37] as well as the possibility of some interesting phases,[35] including an entangled vortex glass state and a melted FLL. Nelson,[35] initially responsible for proposing these peculiar phases, considered only the effects of thermal fluctuations, ignoring the potentially crucial role of disorder; that is, the randomly distributed defects (perhaps oxygen vacancies in the 1:2:3 compounds) responsible for pinning and roughening the vortex lines. This admittedly difficult issue was addressed in the context of Nelson's work by Nattermann and Lipowsky,[36] who provided a quantitative measure of the disorder-induced wandering of an *isolated* flux line, assuming the then best value of ζ_{2+1}. They, furthermore, suggested that Nelson's novel phases would persist. Of course, for most regimes of experimental interest the vortices do not exist by their lonesome, but rather are undeniably part of an elastic, though highly anisotropic FLL. To address properly the phenomena of giant flux creep in ceramic superconductors, one must understand the collective effects of disorder upon the vortex lattice. In the long-wavelength limit, the most natural description relies upon continuum elasticity theory with a 2D disortion field $\mathbf{u}(\mathbf{r})$ describing the local displacement of the FLL in 3D space, a collection of elastic moduli, and a random pinning potential $V_{\rm pin}(u)$ meant to mimic the FLL interaction with defects. A moment's reflection reveals that, for quenched uncorrelated disorder, we have yet another specific application ($n=2$, $d'=3$) of the general ideas proposed herein.

Feigel'man, Geshkenbein, Larkin, and Vinokur,[38] motivated by the controversial experiments indicating giant flux creep in the high-T_c materials, have further developed the Anderson concept of the flux bundle using as an essential ingredient in their analysis the notion of impurity-induced manifold roughening.[55] Operating under the assumption of Bean's critical-state scenario, they find several different regimes of collective flux creep in which the bundle activation barrier U depends on the current j via a power law

$$U(j)\sim j^{-\alpha}\ ,$$

where the exponent α is determined uniquely by the roughening exponent $\zeta_{3,2}$. In particular, for the circumstance special to these ceramic materials in which vortex creep is observed at currents very small compared to the critical current ($j \ll j_c$), they predict

$$\alpha=(1+2\zeta)/(2-\zeta)\ .$$

Other characteristic properties of the bundle diverge similarly as $j\to 0$, including the average bundle size

$$r(j)\sim j^{-1/(2-\zeta)}\ ,$$

as well as the amplitude of the hopping distance

$$u_{\rm hop}(j)\sim j^{-\zeta/(2-\zeta)}\ .$$

Recalling our central result for diverse manifolds in random media,

$$\zeta_{d',n}=2(4-d')/(8+n)$$

(see Sec. III), the functional renormalization group provides us with an estimate for the relevant exponent $\zeta_{3,2}=\frac{1}{5}$ so that $\alpha=\frac{7}{9}$.

As pointed out by Feigel'man and coworkers,[38] the power-law decay of the barrier height has rather startling consequences. Namely, when the superconductor is exposed to a magnetic field giving rise to an Abrikosov FLL, the Bean's critical state is formed on a short time scale τ. At some much later time t, flux creep has been responsible for the decay of the persistent current down to the value $j(t)$, determined implicitly by

$$U(j(t))=T\ln(t/\tau)\ ,$$

where T is the temperature and, with typical laboratory observation time scales, the logarithm is usually of the order ~ 10–30. Because of the power law decay, $U\sim j^{-\alpha}$, it is evident that the current decreases rather rapidly with temperatures and logarithmically in time. These findings are not inconsistent with the results of magnetic measurements of critical currents in which a fast drop of $j(t,T)$ with T increasing and large relaxation effects were noted.

Lastly, Feigel'man, Geshkenbein, Larkin, and Vinokur have managed to construct an energy argument that permits them to relate $\zeta_{d',n}$ with the same n, but different d'. Supplementing their analysis with the known value $\zeta_{1,1}=\frac{2}{3}$ they obtain an explicit prediction for the general case. In fact, the roughening exponent they retrieve is none other than *our fundamental result*, providing strong independent support for the validity of the functional RG estimate. As part of the package, they also find a superuniversal $\beta_c=\frac{1}{2}$.

VI. SUMMARY

The general subject discussed in this paper has been the scaling properties of diverse manifolds in quenched correlated random media. A unified treatment, based on the consideration of an n-dimensional distortion field $\mathbf{z}(\mathbf{x})$ of d'-dimensional support, was proposed and an approximate functional renormalization group applied to estimate $\zeta_{d',n}=2(4-d')/(8+n)$, the sole critical index of interest. An impressive collection of physical problems proved to be nothing but special cases. Starting with domain-wall roughening due to pinning impurities, we considered next the wandering of directed polymers in a disordered environment. The latter, an optimization problem exhibiting an ultrametric free energy landscape, may be relevant in explaining the structures of river basin deltas, capillary networks and neuronal arrays and is of great theoretical interest because it can potentially elucidate many difficult points of spin-glass phenomena. Moreover, because of a transformation via the stochastic Burgers equation, the directed polymer problem is formally identical to the dynamic scaling properties of the surfaces of Eden clusters and ballistic deposits. Using the Derrida-Griffiths real-space renormalization group, we managed, at least for the case of directed polymers, a check on the functional RG estimates. Finally, we considered the effects of weak disorder upon Abrikosov flux lattices in the new ceramic superconductors. A recently proposed theory of collective flux creep, due to Feigel'man *et al.*, have further explored these ideas and managed to retrieve the functional RG value via independent means.

ACKNOWLEDGMENTS

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Comment on "Growth in a Restricted Solid-on-Solid Model"

Recently, Kim and Kosterlitz[1] (KK) reported rather interesting results concerning their simulation of ballistic deposition upon a $(d-1)$-dimensional substrate. More specifically, their data led them to conjecture that at short times, the interfacial width, a measure of the surface roughness, grows as t^{β} with the critical exponent $\beta=1/(d+1)$. As ballistic deposition and Eden cluster growth[2] appear to be in the same universality class and a mapping via the stochastic Burgers equation ties them rather intimately to the physics of directed polymers in random media,[3] the KK conjecture clearly has very strong implications for an impressive variety of problems. Prior to KK, the most precise investigation was that of Wolf and Kertész[2] (WK) on Eden clusters, which yielded their own conjecture $\alpha=2\beta/(\beta+1)=1/d$, for the saturation exponent. Analytic work by Halpin-Healy[4] (and Nattermann), using a functional renormalization-group (RG) treatment of diverse manifolds subject to quenched disorder, retrieved all known exact results, including cases of correlated randomness, for directed polymers in two dimensions and, furthermore, gave $\zeta_{2+1}=3/5$ for the wandering exponent in $d=3$, lending some support to the WK value there.

In this Comment, we present further evidence of a decreasing many-dimensional directed polymer roughening exponent ζ. Using real-space RG (RSRG) methods introduced by Derrida and Griffiths,[5] we calculate the energy fluctuation index $\omega=2\zeta-1$ for directed polymers, which the Burgers equation permits us to identify as the short-time exponent β of ballistic deposition and Eden growth. As a practical matter, the RSRG procedure in the present context amounts to renormalizing, iteratively, a probability distribution for the random bond energies and following the evolution of its width. After many RG transformations, the width renormalization factor approaches an asymptotic value, $\lambda=2^{\omega}$, that depends solely on the spatial rescaling factor b of a given hierarchical lattice. For these lattices, the RSRG method is exact, though here numerically implemented, and we plot our results in Fig. 1. Using the standard, but approximate, correspondence between b-branch hierarchical and d-dimensional hypercubic lattices, $b=2^{d-1}$, we connect to the physically interesting cases. Regarding the dimensionality dependence of the critical indices, the general trend is clear. For $d=1$, the exact value $\omega=\frac{1}{2}$ is recaptured,[1,5] followed by a steady decline toward zero as d is increased, raising further doubts concerning the concept of stochastic superuniversality. As shown previously by Derrida and Griffiths, the result for $d=2$ is a bit below the known exponent $\frac{1}{3}$. The great interest, of course, is for higher dimensionalities. There we find that the RSRG yields $\omega(d=3)\approx 0.204$, a value that is consistent with, though slightly greater than, the functional RG calculation (and WK prediction) of $\frac{1}{5}$, but certainly less than the KK conjecture $\frac{1}{4}$.

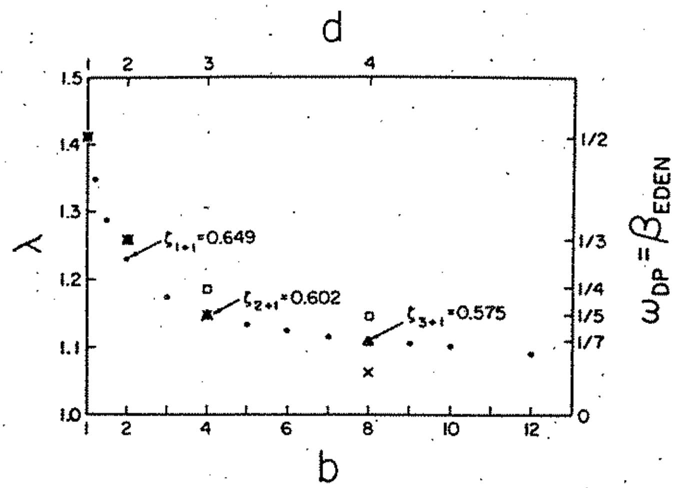

FIG. 1. Results (●) following from a RSRG analysis of the directed polymer problem on generalized hierarchical lattices, extending the method of Derrida and Griffiths. Plotted is the width renormalization factor λ for b-branch hierarchical lattices. The energy fluctuation index for directed polymers is given by $\omega=\ln\lambda/\ln 2$, which in turn is equivalent to the short-time exponent β characterizing roughness of a growing Eden surface. The relation $b=2^{d-1}$ permits explicit, albeit approximate, translation of these results to the more familiar d-dimensional hypercubic lattices. Hence, we obtain new estimates for the wandering exponent ζ, whose numerical values are indicated. For comparison, we include the functional RG work (×) of Ref. 4, as well as the conjectures of KK (□) and WK (△).

Naturally, were the RSRG result yet another underestimate, more credence is lent to the ideas of KK. Subject to the same caveat, we find similar, though still somewhat poorer, agreement with the WK value $\omega(d=4)=\frac{1}{7}$. Finally, we emphasize that an $\epsilon=d-1$ expansion of RSRG,[5] while potentially exact, apparently disagrees with all conjectures to date.

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